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THE CONSTRUCTION OF FILTERS IN NON-LINEAR DETERMINISTIC SYSTEMS†

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The method of residuals (see, e.g. [1–31]) is used to solve the problem of estimation when both object and observations involve noise, and the input determination problem [3–5] is considered. These estimation problems are solved by minimizing a certain functional, and this in turn involves solving a boundary-value problem at each instant of time. Depending on the recurrent method used to solve the relevant family of boundary-value problems, one obtains different representations of optimal non-linear filters for the estimated quantities. The choice of a specific representation depends on the degree to which the object with whose help the filter is being designed is well conditioned. A locally optimal filter of a design similar to that of filters for linear problems is constructed.

The method of residuals has been used to estimate the states and parameters of systems with noisy observations [6].

1. DESCRIPTION OF THE STATE AND INPUT

Consider a system of differential equations

$$X' \doteq f(s, X, \upsilon) \tag{1.1}$$

with observations

$$y(s) \doteq \varphi(s, X(s), \upsilon(s)), \quad t_0 \leqslant s \leqslant t \tag{1.2}$$

The prime denotes differentiation with respect to s; X and y are column vectors with n and m components, respectively, the equations of motion depend on an input $\upsilon = \upsilon(s)$ of dimensions k, and the approximate equality signs in (1.1) and (1.2) indicate the presence of unknown noise in the object and in the observations. The problem is to estimate the input $\upsilon(s)$, $t_0 \leqslant s \leqslant t$, on the basis of observations over the interval $[t_0, t]$ and an estimate of the state $X(s)$.

It is assumed that the functions $f(s, x, \upsilon)$, $\varphi(s, x, \upsilon)$, and indeed all functions in this paper, are such that all subsequent operations based on the assumption that the solutions of systems of differential equations can be extended as functions of time and are differentiable with respect to the initial data and the parameters are allowed. These requirements are met, for example, by assuming as usual that f and φ are smooth functions and that f is of bounded growth.

Let us introduce a system with a control $u = (\upsilon, w)$

$$X' = f(s, X, \upsilon) + D(s)w \tag{1.3}$$

and a functional defined on the solutions of system (1.3)

$$J = \frac{1}{2}(X(t_0) - \bar{x})^T P(X(t_0) - \bar{x}) + \frac{1}{2}\int_{t_0}^{t} [(y(s) - \varphi(s, X, \upsilon(s)))^T Q(s)(y(s) - \varphi(s, X(s), \upsilon(s)) +$$
$$+(\upsilon(s) - \bar{\upsilon}(s))^T S(s)(\upsilon(s) - \bar{\upsilon}(s)) + w^T(s) R(s) w(s)] ds \tag{1.4}$$

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In both (1.3) and (1.4) w is a column vector with r components and D is an $n \times r$ matrix, the $n \times n$ matrix P, the $r \times r$ matrix R and the $k \times k$ matrix S are positive definite, the $m \times m$ matrix Q is positive semi-definite, $\bar{x}$ is a known vector. The function $\bar{\upsilon}(s)$, like $\bar{x}$ for the state $X(t_0)$, is an a priori estimate of the input $\upsilon(s)$. The choice of the matrices D, Q and R depends on the available information about the structure and strength of the noise and P and S depend on the a priori information about the initial data and the input.

An estimate of the state $X(s)$, $t_0 \leqslant s \leqslant t$, based on observations over the interval $[t_0, t]$ will be denoted by $x_{\hat{s/t}}$ ($x_{\hat{t/t}}$ will be denoted by $x_{\hat{t}}$). We shall seek it as an optimal trajectory in the sense that it minimizes the functional (1.4) on solutions of system (1.3).

Note that the controls υ and w have different physical meanings: while an optimal control w ensures that the residual in system (1.1) will be a minimum (in particular, when Eqs (1.1) hold exactly we put D, R and w equal to zero), an optimal υ is an estimate of the unknown input.

2. RECURRENT SOLUTION OF OPTIMUM PROBLEMS

The following optimal control problem is sufficiently general for the type of problems being considered here

$$X' = f(s, X, u) \tag{2.1}$$

$$J = \frac{1}{2}(X(t_0) - \bar{x})^T P(X(t_0) - \bar{x}) + \int_{t_0}^{t} f_0(s, X(s), u(s))ds \to \min \tag{2.2}$$

The minimum in (2.2) extends over a certain class of admissible controls. Let us assume that the functions $f(s, x, \dot{u})$ and $f_0(s, x, u)$ and the constraints on u are such that a control that maximizes the Pontryagin function

$$h(s, x, p, u) = p^T f(s, x, u) - f_0(s, x, u) \to \max$$

can be expressed as a sufficiently smooth function

$$u = u(s, x, p) \tag{2.3}$$

The superposition $h(s, x, p, u)$ of the Pontryagin function with (2.3) is a Hamiltonian function for problem (2.1), (2.2), which we denote by $H(s, x, p)$. Necessary conditions for an optimum in problem (2.1), (2.2) can be expressed as a boundary-value problem

$$X' = f(s, X, u(s, X, p)) = H_p(s, X, p) \tag{2.4}$$

$$p' = -f_x^T(s, X, u(s, X, p))p + f_{0_x}(s, X, u(s, X, p)) = -H_x(s, X, p)$$

$$p(t_0) = P(X(t_0) - \bar{x}), \quad p(t) = 0 \tag{2.5}$$

Here we have used the following notation (which will be used constantly below). Let $y(x)$ be a scalar function and let $z(x) = (z_1(x), \ldots, z_l)$ be a vector-valued function of k variables $x = (x_1, \ldots, x_k)$. Then y_x denotes the column vector (gradient) with components $\delta y/\delta x_j$, $j = 1, \ldots, k$, and z_x the $l \times k$ matrix (Jacobian) with element $\delta y/\delta x_j$, $j = 1, \ldots, k$, $i = 1, \ldots, l$, at the intersection of the ith row and the jth column.

On the assumption that the optimum problem (2.1), (2.2) is solvable (in connection with the estimation problem we shall again denote the solution by $x_{\hat{s|t}}$, $t_0 \leqslant s \leqslant t$, and call it an estimate), and that the boundary-value problem (2.4), (2.5) has a unique solution, the solution will be the component X of the solution of the latter problem.

As t varies over an interval $[t_0, t_0 + T]$, the problem of finding $x_{\hat{s/t}}$ (even for fixed s) involves solving new boundary-value problems for each t. This may be avoided by using a recurrent method to solve a certain family of boundary-value problems (2.7), and this in turn leads to the construction of a filter, e.g. for $x_{\hat{t0/t}}$ or for $x_{\hat{t}}$.

The following theorem yields a t-recurrent method for determining an estimate $\hat{x_{t_0/t}}$ which is the left end of an optimal trajectory in problem (2.1), (2.2) (and at the same time the left end of $X(s)$ in the boundary-value problem (2.4), (2.5)).

Let $X(s; x), p(s; x)$ be the solution of the Cauchy problem for system (2.4) with initial data

$$X(t_0) = x, \quad p(t_0) = P(x - \bar{x}) \tag{2.6}$$

One condition of Theorem 1 is that the matrix $p_x(t; x), t \in [t_0, t_0 + T]$, be non-singular. We note that since $p_x(t_0; x) = P$ is invertible, the matrix $p_x(t; x)$ is invertible for all t sufficiently close to t_0.

Theorem 1. Suppose that for $t \in [t_0, t_0 + T]$ the matrix $p_x^{-1}(t; x)$ exists. Then the estimate $\hat{x_{t0/t}}$ is a solution of the Cauchy problem

$$\frac{dx^{\wedge}_{t_0/t}}{dt} = -p_x^{-1}(t; x^{\wedge}_{t_0/t}) f_{0_x}(t, X(t; x^{\wedge}_{t_0/t}), u(t, X(t; x^{\wedge}_{t_0/t}, 0)), \quad x^{\wedge}_{t_0/t_0} = \bar{x} \tag{2.7}$$

where $X(s; x)$ (together with $p(s; x)$) is a solution of the Cauchy problem (2.4), (2.6); $p_x(s; x)$ (together with $X_x(s; x)$) is a solution of the Cauchy problem for the variational system

$$\begin{aligned} X_x' &= H_{px}(s, X(s;x), p(s;x))X_x + H_{pp}(s, X(s;x), p(s;x))p_x \\ p_x' &= -H_{xx}(s, X(s;x), p(s;x))X_x - H_{xp}(s, X(s;x), p(s;x))p_x \\ X_x(t_0;x) &= I_{n\times n}, \quad p_x(t_0;x) = P \end{aligned} \tag{2.8}$$

The proof proceeds by differentiation with respect to t of the identity

$$p(t; x^{\wedge}_{t_0/t}) \overset{t}{\equiv} 0$$

and of the obvious equality for $t = t_0$: $X(t_0) = \hat{x_{t_0/t_0}} = \bar{x}$.

Systems (2.4), (2.6) and (2.8) are used to evaluate the right-hand side of (2.7) when a numerical method is being used.

To illustrate, let us consider Euler's method. Let $x^{(k)}$ be an approximation for $\hat{x_{t_0/t_k}}$. Then $x^{(k+1)}$ is found as follows. Solve systems (2.4) and (2.8) in the interval $[t_0, t_k]$ with initial data

$$X(t_0)x^{(k)}, \quad p(t_0 = P(x^{(k)} - \bar{x}), \quad X_x(t_0) = I, \quad p_x(t_0) = P$$

As a result one finds $X(t_k; x^{(k)}), p_x(t_k; x^{(k)})$, and finally

$$x^{(k+1)} = x^{(k)} + \Delta t[-p_x^{-1}(t_k; x^{(k)}) f_{0_x}(t_k, X(t_k; x^{(k)}), u(t_k, X(t_k; x^{(k)}), 0))]$$

Thus, to integrate system (2.7) one has to integrate the systems shown above over time intervals $[t_0, t_k]$ of increasing length.

Of course, after finding $\hat{x_{t_0/t}}$ an optimal trajectory $\hat{x_{s/t}}$ of problem (2.1), (2.2) is found as part of the solution $X(s; \hat{x_{t_0/t}})$ of system (2.4), and an optimal control $\hat{u_{s/t}}$ is determined using (2.3)

$$u^{\wedge}_{s/t} = u(s, X(s; x^{\wedge}_{t_0/t}), p(s; x^{\wedge}_{t_0/t})) = u(s, x^{\wedge}_{s/t}, p^{\wedge}_{s/t})$$

We will now consider an equation for $x_t^{\wedge} = \hat{x_{t/t}}$ (which we shall call the filter equation).

Theorem 2. The filter equation for $x_t^{\wedge}$ is

$$dx_t^{\wedge} / dt = f(t, x_t^{\wedge}, u(t, x_t^{\wedge}, 0)) - K(t; x^{\wedge}_{t_0/t}) f_{0_x}(t, x_t^{\wedge}, u(t, x_t^{\wedge}, 0)), \quad x^{\wedge}_{t_0} = \bar{x} \tag{2.9}$$

where $K(s; x)$ is a solution of the Cauchy problem

$$K' = KH_{xx}(s, X(s; x), p(s; x))K + KH_{xp}(s, X(s; x), p(s; x)) + \\ + H_{px}(s, X(s; x), p(s; x))K + H_{pp}(s, X(s; x), p(s; x)) \tag{2.10}$$
$$K(t_0) = P^1$$

Proof. Equation (2.9) with $K(s; x) = X_x(s; x)p_x^{-1}(s; x)$ is derived from the identity $x_t^{\wedge} = X(t; x_{t_0/t}^{\wedge})$, by using (2.7). Equation (2.10) is obtained by differentiating the relationship $K = X_x p_x^{-1}$.

Because of (2.10), Eq. (2.9) yields a slightly more economical estimation algorithm than (2.7), (2.4), (2.6) and (2.8). Indeed, knowing $x_t^{\wedge}$ and $K(t; x_{t_0/t}^{\wedge})$, one can determine $x_{t\Delta/t}^{\wedge}$ from (2.9) by, for example Euler's method. One then integrates (2.4) in the reverse direction with initial data $X(t + \Delta t) = x_{t+\Delta t}^{\wedge}$, $p(t + \Delta t) = 0$. The resulting solution $X(s)$, taken over the interval $[t_0, t_0 + \Delta t]$, obtaining $K(t + \Delta r; x_{t_0/t+\Delta t}^{\wedge})$, and this, together with the already known estimate $x_{t+\Delta t}^{\wedge}$ enables us to proceed to the next step of the numerical integration of the filter (2.9).

We shall now consider the solution of the problem in Section 1. Problem (1.3), (1.4) is of the class represented by problem (2.1), (2.2) and may therefore be tackled using the results of this section. Suppose that by the maximum condition $\upsilon = \upsilon(s, x, p)$, $w = w(s, x, p)$. Setting up appropriate systems for X and p, one can then use either Eq. (2.7) for $x_{t_0/t}^{\wedge}$ (in which case one needs systems for X_x and p_x) or Eqs (2.9) for $x_t^{\wedge}$ and (2.10) for $K(t; x_{t_0/t}^{\wedge})$. Having obtained, say, $x_t^{\wedge}$, one finds $x_{s/t}^{\wedge}$, $p_{s/t}^{\wedge}$, $t_0 \leqslant s \leqslant t$ as solutions of a Cauchy problem for a system like (2.4) with initial data at the right end of the interval $[t_0, t]$: $X(t) = x_t^{\wedge}$, $p(t) = 0$. This solution yields both an optimal estimate $x_{s/t}^{\wedge}$ of the state $X(s)$ and an optimal estimate for the input $\upsilon(s)$

$$\upsilon_{s/t}^{\wedge} = \upsilon(s, x_{s/t}^{\wedge}, p_{s/t}^{\wedge}), \quad t_0 \leqslant s \leqslant t \tag{2.11}$$

based on observations over the interval $[t_0, t]$.

Let us derive the filter equations for estimating the state of the system in the problem

$$X' \doteq f(s, X), \quad y(s) \doteq C(s)X(s) \tag{2.12}$$

where, unlike (1.1) and (1.2), the functions f and φ are independent of υ and, to simplify matters, the observations are assumed to be linear. In that case, only the vector w remains of the required control in the optimal control problem (1.3), (1.4). The maximum condition gives $w = R^{-1}(s)D^T(s)p$, and the necessary conditions can be written as the following boundary-value problem

$$X' = f(s, X) + D(s)R^{-1}(s)D^T(s)p \tag{2.13}$$
$$p' = -f_x^T(s, X)p - C^T(s)Q(s)(y(s) - C(s)X)$$
$$p(t_0) = P(X(t_0) - \bar{x}), \quad p(t) = 0 \tag{2.14}$$

The filter equations (2.9) and (2.10) for this problem are

$$\frac{dx_t^{\wedge}}{dt} = f(t, x_t^{\wedge}) - K(t; x_{t_0/t}^{\wedge})C^T(t)Q(t)(y(t) - C(t)x_t^{\wedge}), \quad x_{t_0}^{\wedge} = \bar{x} \tag{2.15}$$

$$K' = -K\left[C^T(s)Q(s)C(s) - \left\{\frac{\partial f_x^T}{\partial x_j}(s, X(s; x))p(s; x)\right\}\right]K + \\ + f_x(s, X(s; x))K + Kf_x^T(s, X(s; x)) + D(s)R^{-1}(s)D^T(s) \tag{2.16}$$
$$K(t_0; x) = P^{-1}$$

The expression in braces in (2.16) is the $n \times n$ matrix whose columns are $(\partial f_x^T/\partial x_j)p$, $j = 1, \ldots, n$.

When there is no noise in the initial object [6], the filter (2.15), (2.16) is simplified: the first equation in (2.13) is independent of p and the last term drops out of (2.16).

In the linear case $X_x(s; x)$, $p_x(s; x)$, $K(s; x)$ does not depend on x. Hence all algorithms are substantially simplified and the resulting equations (2.9) and (2.10) are just a deterministic version of the Kalman–Bucy filter.

Let us consider the recurrent determination of the input in greater detail for the following linear system

$$X' \doteq A(s)X + B(s)\upsilon(s) \tag{2.17}$$

$$y(s) \doteq C(s)X(s) + G(s)\upsilon(s), \quad t_0 \leqslant s \leqslant t \tag{2.18}$$

The problem of estimating the state X and the input υ is related to the problem of minimizing the functional (1.4) (with $\varphi(s, X, \upsilon)$ replaced by $C(s)X(s) + G(s)\upsilon(s)$) along trajectories of the system

$$X' \doteq A(s)X + B(s)\upsilon + D(s)w \tag{2.19}$$

When there are no constraints on υ and the equality in (2.19) is exact, the problem with $D(s) = 0$, $R(s) = 0$ and fixed t reduces to that considered in [8].

Problem (2.19), (1.4) may arise, for example, when system (1.1) is linearized in the neighbourhood of "nominal" values $X(t_0) = \bar{x}$, $\upsilon(s) = \bar{\upsilon}(s)$. The system of equations for the increments ΔX and $\Delta\upsilon$—which we may denote by X and υ without fear of confusion—takes the form of (2.17), (2.18), and the quantities $\bar{x}$, $\bar{\upsilon}$ in the functional (1.4) will be zeros. Note that even when the initial system (1.1) does not involve noise (the equality in (1.1) is exact), the equality in the linear approximation system (2.17) will no longer be exact.

Let us establish the filter equation for problem (2.19), (1.4). The maximum condition yields the equalities

$$\upsilon = F^{-1}(s)(S(s)\bar{\upsilon}(s) + B^T(s)p + G^T(s)Q(s)(y(s) - C(s)x)) \tag{2.20}$$

$$w = R^{-1}(s)D^T(s)p, \quad F = S + G^TQG$$

Taking these relationships into account, we find the filter equations to be

$$dx_t^\wedge / dt = A(t)x_t^\wedge + B(t)\upsilon_t^\wedge + KC^T(t)Q(t)[y(t) - C(t)x_t^\wedge - G(t)\upsilon_t^\wedge], \quad x_{t_0}^\wedge = \bar{x}$$

$$\upsilon_t^\wedge = F^{-1}(t)(S(t)\bar{\upsilon}(t) + G^T(t)Q(t)(y(t) - C(t)x_t^\wedge))$$

$$dK / dt = -KC^T(Q - QGF^{-1}G^TQ)CK + (A - BF^{-1}G^TQC)K +$$

$$+K(A^T - C^TQGF^{-1}B^T) + BF^{-1}B^T + DR^{-1}D^T, \quad K(t_0) = P^{-1}$$

To look for an estimate $x_{s/t}^\wedge$, $t_0 \leqslant s \leqslant t$ (at the same time determining $p_{s/t}^\wedge$ also), one must solve the following system in the reverse direction

$$X' = A(s)X + B(s)\upsilon(s, X, p) + D(s)R^{-1}(s)D^T(s)p, \quad X(t) = x_t^1$$

$$p' = -A^T(s)p - C^T(s)Q(s)(y(s) - C(s)X - G(s)\upsilon(s, X, p)), \quad p(t) = 0$$

where υ is replaced by the function of (2.20). By (2.11) and (2.20), and estimate $\upsilon_{s/t}^\wedge$ for the input is found using the relation

$$\upsilon_{s/t}^\wedge = F^{-1}(s)(S(s)\bar{\upsilon}(s) + B^T(s)p_{s/t}^\wedge + G^T(s)Q(s)(y(s) - C(s)x_{s/t}^\wedge))$$

Remark 1. The last term in (2.2) may be any non-linear function which is strictly convex in $X(t_0)$.

Remark 2. A previous publication [6] investigated the case in which there is no a priori information about the initial data (i.e. $P = 0$) and the case in which some of the components of the initial vector are known exactly. These problems can also be generalized.

3. LOCALLY OPTIMAL FILTERS

As already pointed out, implementation of the filter (2.9), (2.10) for non-linear systems involves repeated calculations. These may be avoided by the device of locally optimal estimation, that is to say, successive solution of local optimal problems in which the estimate over an interval $[t, t+\Delta t]$ is derived by minimizing the functional

$$\frac{1}{2}(X(t)-x_t^\vee)^T P(t)(X(t)-x_t^\vee)+\int_t^{t+\Delta t} f_0(s, X(s), u(s))ds \tag{3.1}$$

where $x_t^\vee$ is the estimate (we have introduced the notation $x_t^\vee$ for the new estimate) obtained up to time t and $P(t)$ a specially constructed positive definite matrix, $P(t_0) = P$.

We will now derive an equation for $x_t^\vee$ and $L(t) = P^{-1}(t)$. Let $t_{k+1} = t_k + \Delta t$; $x_t^\vee$, L_k be approximations of $x_{tk}^\vee$, $L(t_k)$ $(k = 0, 1, \dots)$, respectively. Solving the minimization problem (3.1) over the interval $[t_0, t_0 + \Delta t]$, we see that the optimal estimate $x_{t_1}^\vee$ is equal to $x_1^\vee + O(\Delta t^2)$, where

$$x_{t_0}^\vee = x_0^\vee = \bar{x}, \quad L(t_0) = L_0 = P^{-1}$$

$$x_1^\vee = x_0^\vee + [f(t_0, x_0^\vee, u(t_0, x_0^\vee, 0)) - L_0 f_{0_x}(t_0, x_0^\vee, u(t_0, x_0^\vee, 0))]\Delta t$$

The following equality holds for the gain matrix $K(t_1; x_{t_0/t_1}^\vee)$(see (2.10))

$$\begin{aligned}K(t_1; x_{t_0/t_1}^\vee) = L_0 + [&L_0 H_{xx}(t_0, X(t_0; x_{t_0/t_1}^\vee), p(t_0; x_{t_0/t_1}^\vee))L_0 + \\ &+ L_0 H_{xp}(t_0, X(t_0; x_{t_0/t_1}^\vee), p(t_0; x_{t_0/t_1}^\vee)) + H_{px}(t_0, X(t_0; x_{t_0/t_1}^\vee) \\ &p(t_0; x_{t_0/t_1}^\vee))L_0 + H_{pp}(t_0, X(t_0; x_{t_0/t_1}^\vee), p(t_0, x_{t_0/t_1}^\vee))]\Delta t + O(\Delta t^2)\end{aligned}$$

Since $X(t_0; x_{t_0/t_1}^\vee) = x_0^\vee + O(\Delta t)$ and $p(t_1; x_{t_0/t_1}^\vee) = 0$, so that $p(t_0; x_{t_0/t_1}^\vee) = O(\Delta t)$, it follows that

$$K(t_1; x_{t_0/t_1}^\vee) = L_1 + O(\Delta t^2)$$

where

$$\begin{aligned}L_1 = L_0 + [&L_0 H_{xx}(t_0, x_0^\vee, 0)L_0 + H_{px}(t_0, x_0^\vee, 0)L_0 + \\ &+ L_0 H_{xp}(t_0, x_0^\vee, 0) + H_{pp}(t_0, x_0^\vee, 0)]\Delta t\end{aligned}$$

Having calculated $x_1^\vee$ and L_1, we proceed to estimate over the interval $[t_1, t_1 + \Delta t]$, taking the last term in (3.1) to be $(X(t_1) - x_1^\vee)^T L^{-1}{}_1 (X(t_1) - x_1^\vee)$ and integrating from $t_1 + \Delta t$. Proceeding as before, we obtain a sequence

$$x_{k+1}^\vee = x_k^\vee + [f(t_k, x_k^\vee, u(t_k, x_k^\vee, 0)) - L_k f_{0_x}(t_k, x_k^\vee, u(t_k, x_k^\vee, 0))]\Delta t$$

$$\begin{aligned}L_{k+1} = L_k + [&L_k H_{xx}(t_k, x_k^\vee, 0)L_k + L_k H_{xp}(t_k, x_k^\vee, 0) + \\ &+ H_{px}(t_k, x_k^\vee, 0)L_k + H_{pp}(t_k, x_k^\vee, 0)]\Delta t\end{aligned}$$

Letting Δt tend to zero, we obtain a locally optimal filter, expressed as a system of differential equations

$$dx_t^\vee / dt = f(t, x_t^\vee, u(t, x_t^\vee, 0)) - Lf_{0_x}(t, x_t^\vee, u(t, x_t^\vee, u(t, x_t^\vee, 0)), \quad x_{t_0}^\vee = \bar{x} \tag{3.2}$$

$$\begin{aligned}dL/dt = &LH_{xx}(t, x_t^\vee, 0)L + LH_{xp}(t, x_t^\vee, 0) + \\ &+ H_{px}(t, x_t^\vee, 0)L + H_{pp}(t, x_t^\vee, 0), \quad L(t_0) = P^{-1}\end{aligned} \tag{3.3}$$

We will show that if $f_0(s, x, u)$ is convex with respect to the variables (x, u) and strictly convex with respect to u, then the solution $L(t)$ of Eq. (3.3) is a positive definite matrix. Indeed, in view of the equality $H_{px} = H_{px}^T$ it will suffice to show that the matrices $H_{xx}(t, x, 0)$ and $-H_{xx}(t, x, 0)$ are positive semi-definite.

Let us evaluate $-H_{xx}(t, x, 0)$ and $H_{xx}(t, x, 0)$ on the assumption that there are no constraints on the control u. We have

$$-H_{xx} = -\left\{\frac{\partial f_x^T}{\partial x_j} p\right\} - \left\{\frac{\partial f_x^T}{\partial u_k} p\right\} u_x + f_{0_{xx}} + f_{0_{xu}} u_x$$

Hence, for $p = 0$

$$-H_{xx}(t,x,0) = f_{0_{xx}}(t,x,u(t,x,0)) + f_{0_{xu}}(t,x,u(t,x,0))u_x(t,x,0) \tag{3.4}$$

Now the maximum condition implies the identity $f_u^T p - f_{0_u} \equiv 0$. Differentiating with respect to x, we get

$$\left\{\frac{\partial f_u^T}{\partial x_j} p\right\} + \left\{\frac{\partial f_u^T}{\partial u_k} p\right\} u_x - f_{0_{ux}} - f_{0_{uu}} u_x = 0$$

whence, putting $p = 0$, we obtain

$$u_x(t,x,0) = -f_{0_{uu}}^{-1}(t,x,u(t,x,0)) f_{0_{ux}}(t,x,u(t,x,0)) \tag{3.5}$$

Substituting (3.5) into (3.4) and taking the equality $f_{0_{xu}} = f_{0_{ux}}^T$ into consideration, we obtain

$$\begin{aligned} -H_{xx}(t,x,0) &= f_{0_{xx}}(t,x,u(t,x,0)) - \\ &- f_{0_{ux}}^T(t,x,u(t,x,0)) f_{0_{uu}}^{-1}(t,x,u(t,x,0)) f_{0_{ux}}(t,x,u(t,x,0)) \end{aligned} \tag{3.6}$$

It can be shown that, under our assumptions concerning f_0, the matrix on the right of (3.6) is indeed positive semi-definite.

Similar manipulations yield the equality

$$H_{pp}(t,x,0) = f_u(t,x,u(t,x,0)) f_{0_{uu}}^{-1}(t,x,u(t,x,0)) f_u^T(t,x,u(t,x,0))$$

which, together with (3.6), proves that the gain matrix $L(t)$ is indeed positive definite when there are no constraints on the control.

It can be proved that, even when constraints are imposed on the control (assuming, of course, as before, that the function $u(s, x, p)$ is sufficiently smooth), the matrix $L(t)$ will still be positive definite.

To implement the filter (3.2), (3.3), unlike the optimal filter, it is no longer necessary to perform repeated calculations or to store all a priori information about observations. This filter is of the same dimensions as a linear filter. However, the gain matrix cannot be evaluated in advance: it depends on the current estimate $\check{x}_t$, i.e. in the final analysis, on the observations.

It should be noted that, in the case of a linear system with quadratic objective function, the above procedure of locally optimal estimation yields an optimal estimate.

The locally optimal filter corresponding to the filter (2.15), (2.16) has the form

$$d\check{x}_t / dt = f(t,\check{x}_t) + LC^T(t)Q(t)(y(t) - C(t)\check{x}_t), \quad \check{x}_{t_0} = \bar{x} \tag{3.7}$$

$$\begin{aligned} dL/dt &= -LC^T(t)Q(t)C(t)L + f_x(t,\check{x}_t)L + \\ &+ Lf_x^T(t,\check{x}_t) + D(t)R^{-1}(t)D^T(t), \quad L(t_0) = P^{-1} \end{aligned} \tag{3.8}$$

A construction similar to (3.7), (3.8) was obtained when designing an approximate non-linear filter in the stochastic estimation problem (see [9]).

4. DIFFERENT REPRESENTATION OF FILTERS

For simplicity, we will confine our attention to estimating the state of a system (2.12) in which the filter is defined by Eq. (2.15). The basic object in constructing this filter is the solution $X(s;x), p(s;x)$ of the Cauchy problem for system (2.13) with initial data at the left end of the interval $[t_0, t]$: $X(t_0) = x, p(t_0) = P(x-\bar{x})$.

We shall associate the construction of the filter with another Cauchy problem for system (2.13). Let $X(s;t,x), p(s;t,x)$ denote the solution of (2.13) with initial data at the right end of the interval $[t_0, t]$: $X(t) = x, p(t) = 0$. We obtain the following equation for the desired estimate $x = x_t^\wedge$

$$p(t_0;t,x_t^\wedge) \overset{t}{=} P(X(t_0;t,x_t^\wedge)-\bar{x}) \tag{4.1}$$

which implies the following theorem.

Theorem 3. Suppose that for $t \in [t_0, t_0+T]$ the matrix $(PX_x(t_0;t,x)-p_z(t_0;t,x))^{-1}$ exists. Then an estimate $x_t^\wedge$ of the state $X(t)$ in system (2.12) which is optimal in problem (1.3), (1.4) is a solution of the Cauchy problem

$$dx_t^\wedge / dt = -(PX_x(t_0;t,x_t^\wedge) - p_x(t_0;t,x_t^\wedge))^{-1}(PX_t(t_0;t,x_t^\wedge) -$$
$$-p_t(t_0;t,x_t^\wedge)), \quad x_{t_0}^\wedge = \bar{x} \tag{4.2}$$

where the matrices $X_x(s;t,x)\, p_x(s;t,x)$ satisfy the system of variational equations (2.8) with

$$H(s,x,p) = p^T(f(s,x) + D(s)R^{-1}(s)D^T(s)p) - (y(s)-C(s)x)^TQ(s)(y(s)-C(s)x) \tag{4.3}$$

and initial data

$$X_x(t;t,x) = I, \quad p_x(t;t,x) = 0 \tag{4.4}$$

The vectors $X_t(s;t,x)\, p_x(s;t,x)$ satisfy the system of equations

$$X_t' = H_{px}X_t + H_{pp}p_t, \quad p_t' = -H_{xx}X_t - H_{xp}p_t \tag{4.5}$$

with initial data

$$X_t(t;t,x) = -f(t,x), \quad p_t(t;t,x) = C^T(t)Q(t)(y(t)-C(t)x) \tag{4.6}$$

One condition of Theorem 3 is that the matrix $PX_x(t_0 s;t,x) - p_x(t_0;t,x)$ must be non-singular for $t \in [t_0; t_0+T]$. Note that as the matrix $PX_x(t_0;t_0,x) - p_x(t_0;t_0,x) = P$ is invertible, the same is true for all t sufficiently close to t_0.

It may turn out that the Cauchy problems for system (2.13) with initial data at the left and the right ends of the interval are ill posed (this happens, for example, with long time intervals when the initial system (2.12) is asymptotically stable). In that case a more natural choice for the basic object of the construction is a suitable boundary-value problem.

For example, consider the boundary-value problem for system (2.13) with boundary conditions

$$X(t_0) = x, \quad p(t) = 0 \tag{4.7}$$

Denote the solution by $X(s;t,x), p(s;t,x)$. It should be noted that the notation for the solution always involves only the arguments to be varied, but the meaning may be different: whereas in (4.1), (4.2) $X(s;t,x), p(s;t,x)$ denotes a solution of the Cauchy problem for system (2.13) with $X(t) = x, p(t) = 0$, here the same notation stands for a solution of the boundary-value problem (2.13), (4.7).

Theorem 4. Suppose that for $t \in [t_0; t_0+T]$ the matrix $(P - p_x(t_0;t,x)^{-1})$ exists. Then an optimal estimate $x_{t_0/t}^\wedge$ for the initial state $X(t_0)$ in system (2.12) satisfies the Cauchy problem

$$dx^{\wedge}_{t_0/t}/dt = (P - p_x(t_0;t,x^{\wedge}_{t_0/t}))^{-1}p_t(t_0;t,x^{\wedge}_{t_0/t}), \quad x^{\wedge}_{t_0/t_0} = \bar{x} \tag{4.8}$$

where $p_x(s;t,x)$ (together with $X_x(s;t,x)$) is a solution of the following boundary-value problem for system (2.8), (4.3)

$$X_x(t_0;t,x) = I, \quad p_x(t;t,x) = 0 \tag{4.9}$$

and $p_t(s;t,x)$ (together with $X_t(s;t,x)$) is a solution of the following boundary-value problem for system (4.5), (4.3)

$$X_t(t_0;t,x) = 0, \quad p_t(t;t,x) = C^T(t)Q(t)(y(t) - C(t)X(t;t,x)) \tag{4.10}$$

The proof is obtained by differentiating the following identity with respect to t

$$P(x^{\wedge}_{t_0/t} - \bar{x}) \overset{t}{\equiv} p(t_0;t,x^{\wedge}_{t_0/t})$$

One possible procedure for constructing the filter (4.8) is as follows. Knowing $x^{\wedge}_{t_0/t}$ and a solution $X(s;t,x^{\wedge}_{t_0/t}), p(s;t,x^{\wedge}_{t_0/t})$ find (say, by differential or difference pivotal condensation methods) solutions $X_x(s;t,x^{\wedge}_{t_0/t}), p_x(s;t,x^{\wedge}_{t_0/t})$ and $X_t(s;t,x^{\wedge}_{t_0/t}), p_t(s;t,x^{\wedge}_{t_0/t})$ of the linear boundary-value problems (2.8), (4.3), (4.9) and (4.5), (4.3), (4.10) with $x = x^{\wedge}_{t_0/t}$ for $t_0 \leqslant s \leqslant t$. Then, having selected a stepsize Δt for the time, use Euler's method to derive $x = x^{\wedge}_{t_0/t+\Delta t}$ from (4.8) accurate to within $O(\Delta t^2)$ at each step. Put $\Delta x^{\wedge}_t = x^{\wedge}_{t_0/t+\Delta t} - x^{\wedge}_{t_0/t}$. Then we can write

$$X(s;t+\Delta t,x^{\wedge}_{t_0/t+\Delta t}) = X(s;t,x^{\wedge}_{t_0/t}) + X_t(s;t,x^{\wedge}_{t_0/t})\Delta t +$$
$$+ X_x(s;t,x^{\wedge}_{t_0/t})\Delta x^{\wedge}_t + O(\Delta t^2), \quad t_0 \leqslant s \leqslant t$$

An analogous relationship will hold for $p(s;t+\Delta t,x^{\wedge}_{t_0/t+\Delta t})$. The solution obtained in the interval $[t_0,t]$ may then be continued to the interval $[t_0,t+\Delta t]$, e.g. by solving the Cauchy problem in the short time interval $[t,t+\Delta t]$. This gives $x^{\wedge}_{t_0/t+\Delta t}, X(s;t+\Delta t,x^{\wedge}_{t_0/t+\Delta t}), p(s;t+\Delta t,x^{\wedge}_{t_0/t+\Delta t}), t_0 \leqslant s \leqslant t+\Delta t$, from which one can now proceed to the next step.

To avoid the accumulation of errors, one must take measures (e.g. by Newton's method) to improve the solution of the boundary-value problem (2.13), (2.14) after every few steps.

Note that when there is no noise in system (2.12), the boundary-value problems just considered reduce, thanks to the fact that X is independent of p, to successive Cauchy problems: from left to right for X, and then from right to left for p.

Besides the filter representations already mentioned, there are many others. For example, if the basic object is taken to be a boundary-value problem for (2.13) with boundary conditions for X at the left and right ends, then the original boundary conditions (2.14) yield $2n$ identities for the t-recurrent determination of $x^{\wedge}_{t_0/t}$ and $x^{\wedge}_t$. In principle, any boundary-value problem for (2.13) that is consistent with the original boundary conditions (2.14) may be used. Such a boundary-value problem may contain k, $0 \leqslant k \leqslant 2n$, unknowns, relative to which one can always derive from (2.14) k identities (as functions of t). The result will be a certain representation of the filter.

Let us consider the filter obtained by letting the basic object be the boundary-value problem (2.13), (2.14) itself. In that case $k = 0$ and the solution $X(s;t), p(s;t), t_0 \leqslant s \leqslant t$ depends only on t. Clearly, $x^{\wedge}_{s/t} = X(s;t)$.

Theorem 5. The equation of the filter for $x^{\wedge}_t$ is

$$dx^{\wedge}_t/dt = f(t,x^{\wedge}_t) + X_t(t;t), \quad x^{\wedge}_{t_0} = \bar{x} \tag{4.11}$$

where $X_t(s;t), p(s;t), t_0 \leqslant s \leqslant t$ is a solution of system (4.5), (4.3) with boundary conditions

$$p_t(t_0;t) = PX_t(t_0;t), \quad p_t(t;t) = C^T(t)Q(t)(y(t) - C(t)X(t;t)) \tag{4.12}$$

The proof (as in the previous theorems) follows from the identity $x_t^\wedge \equiv X(t; t)$.

If the criterion is the number of dimensions of the auxiliary system of differential equations needed to formulate the characteristics of the filter, the most rational choice for the basic object is indeed the original problem (2.13), (2.14).

We mention, as an example, that the characteristics of a filter have been formulated on the basis of a solution $X(s; t, c), p(s; t, c)$ of a boundary-value problem for system (2.13) with boundary conditions $p(t_0) = P(X(t_0) - \bar{x}), p(t) = c$ [1, 2]. This problem, unlike those considered here, is not consistent (if $c \neq 0$) with the boundary conditions of the initial problem, and the construction of the filter requires a knowledge not only of X_t, p_t, but also of X_c, p_c.

Remark 3. The question of which filter representation to choose depends on the specific features of the problem. It would seem that the recommendations depend essentially on the exponential dichotomy property of the system.

Here ([6], see also [1–3]) we have used a priori information on the state at the initial time. Such information does not always regularize the estimation problem (an example is the equation $x'' = a^2 x$ for large a). In such cases one naturally appeals to additional information about the boundary conditions. This approach yields different estimation algorithms, which also admit of a variety of filter representations. Of course, the questions touched upon in this remark require special investigation.

5. NUMERICAL EXPERIMENTS

As (1.1) and (1.2), consider the system

$$\begin{aligned} X' &\doteq \lambda X(4 - X^2), \quad \lambda > 0 \\ y(s) &\doteq X(s), \quad t_0 \leqslant s \leqslant t \end{aligned} \tag{5.1}$$

(Eq. (5.1) is the equation for the first approximation of the amplitude in a solution of Van der Pol's equation). Taking $D(s) \equiv 1$, $Q(s) \equiv 1$, $R(s) \equiv 1$ in (1.3) and (1.4), one finds the optimal estimate for $X(s)$ by solving the following boundary-value problem (see Section 2)

$$X' = \lambda X(4 - X^2) + p, \tag{5.2}$$

$$\begin{aligned} p' &= -\lambda(4 - 3X^2)p - (y(s) - X) \\ p(t_0) &= P(X(t_0) - \bar{x}), \quad p(t) = 0 \end{aligned} \tag{5.3}$$

We will write two representations of the filter, deriving from different basic objects. In the first case we use Cauchy problems. The filter will be

$$\begin{aligned} dx^\wedge_{t_0/t} / dt &= p_x^{-1}(t; x^\wedge_{t_0/t})(y(t) - X(t; x^\wedge_{t_0/t})) \\ x_t^\wedge &= X(t; x^\wedge_{t_0/t}) \end{aligned} \tag{5.4}$$

Implementation of this filter involves the solution of initial-value problems (see Section 2) in intervals $[t_0, t]$, $t_0 \leqslant t \leqslant T$, of increasing length for system (5.2) and the variational system

$$\begin{aligned} X_x' &= \lambda(4 - 3X^2)X_x + p_x, \\ p_x' &= (1 + 6\lambda X_p)X_x - \lambda(4 - 3X^2)p_x \end{aligned} \tag{5.5}$$

The second filter derives from the boundary-value problem (5.2), (5.3); it has the form

$$dx_t^\wedge / dt = \lambda x_t^\wedge (4 - (x_t^\wedge)^2) + X_t(t; t), \quad x^\wedge_{t_0/t} = X(t_0; t) \tag{5.6}$$

To implement (5.6) one has to solve an auxiliary linear boundary-value problem (see Section 4) for the system

$$X_t' = \lambda(4 - 3X^2)X_t + p_t, \quad p_t' = (1 + 6\lambda X_p)X_t - \lambda(4 - 3X^2)p_t \tag{5.7}$$

Table 1

Δ	$\lambda = 0.1$	0.5	1.0
	2.310	1.986	1.591
0.1	2.312	2.010	2.000
	2.309	2.010	2.000
	2.320	1.836	1.141
0.5	2.329	2.011	2.000
	2.325	2.010	2.000

in intervals $[t_0, t]$ of increasing length, and a Cauchy problem for (5.2) in the intervals $[t, t + \Delta t]$. Note that the initial-value problem for system (5.2), (5.5) is computationally unstable (even for relatively small $\lambda > 0$, while the boundary-value problem for system (5.7) is stable. Although the exact solutions of the equations of different filters yield the same optimal estimates, the results of numerical integration may differ considerably.

The numerical experiments were carried out with $X(t_0) = 3, \bar{x} = 3(1 + \Delta), 0 \leqslant s \leqslant 1$; the noise in both system and observations was a sum of high-frequency oscillations.

Table 1 gives experimental results for different λ and Δ, compared with the optimal estimate. The optimal estimate $x_t^{\wedge}$ was calculated using highly accurate methods with a small step size in numerical integration. Implementation of the filters (5.4) and (5.6) involved solving Cauchy problems using Euler's method. The linear boundary-value problem for (5.7) was solved by the difference pivotal condensation method. The first row in Table 1 gives results for the filter (5.4) for each Δ value, and the second, for the filter (5.6). The third row presents the optimal estimate.

Analysis of the numerical results showed that for small values of the parameter λ (see $\lambda = 0.1$) both algorithms give similar values of the estimate, irrespective of errors in the system and the observations, or of the closeness of the a priori value $\bar{x}$ to $X(t_0)$. As λ increases (see $\lambda = 0.5$) the instability of the initial problems is amplified, affecting the quality of the numerical solution of (5.4), especially at distant $\bar{x}$ values. For certain λ values (see $\lambda = 1.0$) application of the first filter makes the numerical value of the estimate deviate substantially from the optimal estimate $x_t^{\wedge}$ even over short observation intervals. At the same time, the numerical solution relying on a boundary-value problem yields a nearly optimal estimate.

The optimal estimate $x_{t_0/t}^{\wedge}$ of the initial state $X(t_0)$ in this experiment levelled off quite quickly. In the numerical implementation of the filter, the estimate for $X(t_0)$ may only deteriorate in quality as the observation interval becomes longer, owing to computation errors. The effect is more marked in relation to the numerical integration of the first filter. The influence of computation errors appears even in fairly short intervals if λ is increased.

Thus, in system (5.1), the second filter representation (5.6) is more satisfactory from the point of view of computation.

Note that in all experiments with this example, a locally optimal filter gives an estimate $x_t^{\vee}$ near the optimal estimate $x_t^{\wedge}$.

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